

Bulk Universality for Unitary Matrix Models

M.Poplavskyi

Institute for Low Temperature Physics Ukr. Ac. Sci.,

Lenin ave. 47, Kharkov, Ukraine

E-mail: poplavskiy Mihail@rambler.ru

Abstract

We give a proof of universality in the bulk of spectrum of unitary matrix models, assuming that the potential is globally C^2 and locally C^3 function (see Theorem 1.2). The proof is based on the determinant formulas for correlation functions in terms of polynomials orthogonal on the unit circle. We do not use asymptotics of orthogonal polynomials. We obtain the *sin*-kernel as a unique solution of a certain non-linear integro-differential equation.

Key words: unitary matrix models, local eigenvalue statistics, universality

1 Introduction

In this paper we study a class of random matrix ensembles, known as unitary matrix models. These models are defined by the probability law

$$p_n(U) d\mu_n(U) = Z_{n,2}^{-1} \exp \left\{ -n \operatorname{Tr} V \left(\frac{U + U^*}{2} \right) \right\} d\mu_n(U), \quad (1.1)$$

where $U = \{U_{jk}\}_{j,k=1}^n$ is a $n \times n$ unitary matrix, $\mu_n(U)$ is the Haar measure on the group $U(n)$, $Z_{n,2}$ is the normalization constant and $V : [-1, 1] \rightarrow \mathbb{R}^+$ is a continuous function, called the potential of the model. Denote $e^{i\lambda_j}$ the eigenvalues of unitary matrix U . The joint probability density of λ_j , corresponding to (1.1), is given by (see [1])

$$p_n(\lambda_1, \dots, \lambda_n) = \frac{1}{Z_n} \prod_{1 \leq j < k \leq n} |e^{i\lambda_j} - e^{i\lambda_k}|^2 \exp \left\{ -n \sum_{j=1}^n V(\cos \lambda_j) \right\}. \quad (1.2)$$

To simplify notations, below we write $V(x)$ instead of $V(\cos x)$. Normalized Counting Measure of eigenvalues (NCM) is given by

$$N_n(\Delta) = n^{-1} \# \left\{ \lambda_l^{(n)} \in \Delta, l = 1, \dots, n \right\}, \quad \Delta \subset [-\pi, \pi]. \quad (1.3)$$

The random matrix theory deals with several asymptotic regimes of the eigenvalue distribution. The global regime is centered around weak convergence of NCM (1.3). Global regime for unitary matrix models was studied in [2]. We will use the main result of [2]:

Theorem 1.1 Assume that the potential V of the model (1.1) is a $C^2(-\pi, \pi)$ function. Then:

- there exists a measure $N \in \mathcal{M}_1([-\pi, \pi])$ with a compact support σ , such that NCM N_n converges in probability to N ;

- N has a bounded density ρ ;
- denote $\rho_n := p_1^{(n)}$ the first marginal density, then for any $\phi \in H^1(-\pi, \pi)$

$$\left| \int \phi(\lambda) \rho_n(\lambda) d\lambda - \int \phi(\lambda) \rho(\lambda) d\lambda \right| \leq C \|\phi\|_2^{1/2} \|\phi'\|_2^{1/2} n^{-1/2} \ln^{1/2} n, \quad (1.4)$$

where $\|\cdot\|_2$ denotes L_2 norm on $[-\pi, \pi]$

One of the main topics of local regime is universality of local eigenvalue statistics. Let

$$p_l^{(n)}(\lambda_1, \dots, \lambda_l) = \int p_n(\lambda_1, \dots, \lambda_l, \lambda_{l+1}, \dots, \lambda_n) d\lambda_{l+1} \dots d\lambda_n \quad (1.5)$$

be the l -th marginal density of p_n .

Definition 1.1 We call by the bulk of the spectrum the set

$$\{\lambda \in \sigma : \rho(\lambda) > 0\}, \quad (1.6)$$

where ρ is defined in Theorem 1.1.

The main result of the paper is the proof of the universality conjecture in the bulk of spectrum:

$$\lim_{n \rightarrow \infty} [n\rho_n(\lambda)]^{-l} p_l^{(n)}\left(\lambda + \frac{x_1}{n\rho_n(\lambda)}, \dots, \lambda + \frac{x_l}{n\rho_n(\lambda)}\right) = \det \{S(x_j - x_k)\}_{j,k=1}^l, \quad (1.7)$$

where

$$S(x) = \frac{\sin \pi x}{\pi x}. \quad (1.8)$$

In accordance with (1.7), limiting local distributions of eigenvalues do not depend on potential V in (1.1), modulo some weak condition (see Theorem 1.2). The conjecture of universality of all correlation functions was suggested by Dyson (see [3]) in early 60-th who proved (1.7) - (1.8) for $V(x) = 0$. First rigorous proofs for Hermitian matrix models with non quadratic V appeared only in 90-th. The case of general V which is locally C^3 function was studied in [4]. The case of real analytic potential V was studied in [5], where the asymptotics of orthogonal polynomials were obtained. For unitary matrix models bulk universality was proved for $V = 0$ (see [3]) and in the case of a linear V (see [6]).

To prove the main result we need some properties of the polynomials orthogonal with a variable weight on the unit circle. Consider the system of functions $\{e^{ik\lambda}\}_{k=0}^\infty$ and use for them the Gram-Shmidt procedure in $L_2([-\pi, \pi], e^{-nV(\lambda)})$. For any n we get the system of functions $\{P_k^{(n)}(\lambda)\}_{k=0}^\infty$ which are orthogonal and normalized in $L_2([-\pi, \pi], e^{-nV(\lambda)})$. Since V is even, it is easy to see that all coefficients of these functions are real. Denote

$$\psi_k^{(n)}(\lambda) = P_k^{(n)}(\lambda) e^{-nV(\lambda)/2}. \quad (1.9)$$

Then we obtain the orthogonal in $L_2(-\pi, \pi)$ functions

$$\int_{-\pi}^{\pi} \psi_k^{(n)}(\lambda) \overline{\psi_l^{(n)}(\lambda)} d\lambda = \delta_{kl}. \quad (1.10)$$

The reproducing kernel of the system (1.9) is given by

$$K_n(\lambda, \mu) = \sum_{j=0}^{n-1} \psi_j^{(n)}(\lambda) \overline{\psi_j^{(n)}(\mu)}. \quad (1.11)$$

From (1.10) we obtain, that the reproducing kernel satisfies the relation

$$\int_{-\pi}^{\pi} K_n(\lambda, \nu) K_n(\nu, \mu) d\nu = K_n(\lambda, \mu), \quad (1.12)$$

and from the Cauchy inequality we have

$$|K_n(\lambda, \mu)|^2 \leq K_n(\lambda, \lambda) K_n(\mu, \mu). \quad (1.13)$$

We use also below the determinant form of the marginal densities (1.5) (see [1])

$$p_l^{(n)}(\lambda_1, \dots, \lambda_l) = \frac{(n-l)!}{n!} \det \|K_n(\lambda_j, \lambda_k)\|_{j,k=1}^l. \quad (1.14)$$

In particular,

$$\rho_n(\lambda) = n^{-1} K_n(\lambda, \lambda), \quad (1.15)$$

$$p_2^{(n)}(\lambda, \mu) = \frac{K_n(\lambda, \lambda) K_n(\mu, \mu) - |K_n(\lambda, \mu)|^2}{n(n-1)}. \quad (1.16)$$

The main result of the paper is

Theorem 1.2 Assume that $V(\lambda)$ is a $C^2(-\pi, \pi)$ function, and there exists an interval $(a, b) \subset \sigma$ such that

$$\sup_{\lambda \in (a, b)} |V'''(\lambda)| \leq C_1, \quad \rho(\lambda) \geq C_2, \quad \lambda \in (a, b). \quad (1.17)$$

Then for any $d > 0$ and $\lambda_0 \in [a + d, b - d]$ for K_n defined in (1.11) we have

$$\begin{aligned} \lim_{n \rightarrow \infty} [K_n(\lambda_0, \lambda_0)]^{-1} K_n\left(\lambda_0 + \frac{x}{K_n(\lambda_0, \lambda_0)}, \lambda_0 + \frac{y}{K_n(\lambda_0, \lambda_0)}\right) = \\ = e^{i(x-y)/2\rho(\lambda_0)} \frac{\sin \pi(x-y)}{\pi(x-y)} \end{aligned} \quad (1.18)$$

uniformly in (x, y) , varying on a compact set of \mathbb{R}^2 .

Remark 1.3 It is easy to see that universality conjecture (1.7) follows from Theorem 1.2 by (1.14).

The method of the proof is a version of the method of [4]. An important part of the proof is uniform convergence of ρ_n to ρ in a neighborhood of λ_0 :

Theorem 1.4 Under the assumptions of Theorem 1.2 for any $d > 0$ there exists $C(d) > 0$ such that for any $\lambda \in [a + d, b - d]$

$$|\rho_n(\lambda) - \rho(\lambda)| \leq C(d) n^{-2/9}. \quad (1.19)$$

2 Proof of basic results

Proof of Theorem 1.4. We will use some facts from integral transformations theory (see [7]).

Definition 2.1 Assume that $g(\lambda)$ is a continuous function on the interval $[-\pi, \pi]$. Then its Germglotz transformation is given by

$$F[g](z) = \int_{-\pi}^{\pi} \frac{e^{i\lambda} + e^{iz}}{e^{i\lambda} - e^{iz}} g(\lambda) d\lambda, \quad (2.1)$$

where $z \in \mathbb{C} \setminus \mathbb{R}$.

Inverse transformation is given by

$$g(\mu) = \frac{1}{2\pi} \lim_{z \rightarrow \mu + i0} \Re F[g](z). \quad (2.2)$$

For $z = \mu + i\eta$, $\eta \neq 0$ we set

$$f_n(z) = \int_{-\pi}^{\pi} \frac{e^{i\lambda} + e^{iz}}{e^{i\lambda} - e^{iz}} \rho_n(\lambda) d\lambda. \quad (2.3)$$

Bellow we will derive a "square" equation for f_n . Denote

$$\mathcal{I}_n(z) = \int_{-\pi}^{\pi} V'(\lambda) \frac{e^{i\lambda} + e^{iz}}{e^{i\lambda} - e^{iz}} \rho_n(\lambda) d\lambda. \quad (2.4)$$

Integrating by parts in (2.4), we obtain from (1.5)

$$\begin{aligned} \mathcal{I}_n(z) &= \frac{1}{Z_n} \int V'(\lambda_1) \frac{e^{i\lambda_1} + e^{iz}}{e^{i\lambda_1} - e^{iz}} \prod_{j < k} \left| e^{i\lambda_j} - e^{i\lambda_k} \right|^2 \exp \left\{ -n \sum_{j=1}^n V(\lambda_j) \right\} \prod_{j=1}^n d\lambda_j = \\ &= \frac{1}{nZ_n} \int e^{-nV(\lambda_1)} \frac{d}{d\lambda_1} \left(\frac{e^{i\lambda_1} + e^{iz}}{e^{i\lambda_1} - e^{iz}} \prod_{j < k} \left| e^{i\lambda_j} - e^{i\lambda_k} \right|^2 \exp \left\{ -n \sum_{j=2}^n V(\lambda_j) \right\} \right) \prod_{j=1}^n d\lambda_j. \end{aligned}$$

The integrated term equals 0, because all functions here are 2π -periodic. After differentiation we have the sum of n terms under integral. Denote

$$\begin{aligned} I_0(z) &= \frac{1}{nZ_n} \int \frac{d}{d\lambda_1} \left(\frac{e^{i\lambda_1} + e^{iz}}{e^{i\lambda_1} - e^{iz}} \right) \prod_{j < k} \left| e^{i\lambda_j} - e^{i\lambda_k} \right|^2 \exp \left\{ -n \sum_{j=1}^n V(\lambda_j) \right\} \prod_{j=1}^n d\lambda_j, \\ I_m(z) &= \frac{1}{nZ_n} \int \frac{e^{i\lambda_1} + e^{iz}}{e^{i\lambda_1} - e^{iz}} \prod_{2 \leq j < k \leq n} \left| e^{i\lambda_j} - e^{i\lambda_k} \right|^2 \frac{d}{d\lambda_1} \left| e^{i\lambda_1} - e^{i\lambda_m} \right|^2 \cdot \\ &\quad \cdot \prod_{k \neq m} \left| e^{i\lambda_1} - e^{i\lambda_k} \right|^2 \exp \left\{ -n \sum_{j=1}^n V(\lambda_j) \right\} \prod_{j=1}^n d\lambda_j, \quad m = \overline{2, n}. \end{aligned}$$

From symmetry with respect to λ_j we obtain that all $I_m(z)$, except $I_0(z)$, are equal, hence

$$\mathcal{I}_n(z) = I_0(z) + (n-1) I_2(z).$$

$$\begin{aligned} I_0(z) &= \frac{1}{n} \int_{-\pi}^{\pi} \frac{d}{d\lambda_1} \left(\frac{e^{i\lambda_1} + e^{iz}}{e^{i\lambda_1} - e^{iz}} \right) \rho_n(\lambda_1) d\lambda_1 = \\ &= -\frac{2i}{n} \int_{-\pi}^{\pi} \frac{e^{i\lambda_1} e^{iz}}{(e^{i\lambda_1} - e^{iz})^2} \rho_n(\lambda_1) d\lambda_1 = -\frac{i}{2n} \int_{-\pi}^{\pi} \left(\frac{e^{i\lambda_1} + e^{iz}}{e^{i\lambda_1} - e^{iz}} \right)^2 \rho_n(\lambda_1) d\lambda_1 + \frac{i}{2n}. \end{aligned}$$

To transform I_2 , we use symmetry of $p_2^{(n)}$ ($p_2^{(n)}(\lambda_1, \lambda_2) = p_2^{(n)}(\lambda_2, \lambda_1)$).

$$\begin{aligned}
I_2(z) &= \frac{1}{n} \int \frac{e^{i\lambda_1} + e^{iz}}{e^{i\lambda_1} - e^{iz}} \frac{d}{d\lambda_1} \frac{|e^{i\lambda_1} - e^{i\lambda_2}|^2}{|e^{i\lambda_1} - e^{i\lambda_2}|^2} p_2^{(n)}(\lambda_1, \lambda_2) d\lambda_1 d\lambda_2 = \\
&= \frac{i}{n} \int \frac{e^{i\lambda_1} + e^{iz}}{e^{i\lambda_1} - e^{iz}} \frac{e^{i\lambda_1} + e^{i\lambda_2}}{e^{i\lambda_1} - e^{i\lambda_2}} p_2^{(n)}(\lambda_1, \lambda_2) d\lambda_1 d\lambda_2 = \\
&= \frac{i}{2n} \int \left(\frac{e^{i\lambda_1} + e^{iz}}{e^{i\lambda_1} - e^{iz}} - \frac{e^{i\lambda_2} + e^{iz}}{e^{i\lambda_2} - e^{iz}} \right) \frac{e^{i\lambda_1} + e^{i\lambda_2}}{e^{i\lambda_1} - e^{i\lambda_2}} p_2^{(n)}(\lambda_1, \lambda_2) d\lambda_1 d\lambda_2 = \\
&= -\frac{i}{2n} \int \frac{2(e^{i\lambda_1} + e^{i\lambda_2})e^{iz}}{(e^{i\lambda_1} - e^{iz})(e^{i\lambda_2} - e^{iz})} p_2^{(n)}(\lambda_1, \lambda_2) d\lambda_1 d\lambda_2 = \\
&= \frac{i}{2n} - \frac{i}{2n} \int \frac{e^{i\lambda_1} + e^{iz}}{e^{i\lambda_1} - e^{iz}} \frac{e^{i\lambda_2} + e^{iz}}{e^{i\lambda_2} - e^{iz}} p_2^{(n)}(\lambda_1, \lambda_2) d\lambda_1 d\lambda_2.
\end{aligned}$$

Therefore, from (1.5) and (1.14), we obtain

$$\mathcal{I}_n(z) = \frac{i}{2} - \frac{i}{2} f_n^2(z) - \frac{i}{n^2} \int |K_n(\lambda_1, \lambda_2)|^2 \frac{(e^{i\lambda_1} - e^{i\lambda_2})^2 e^{2iz}}{(e^{i\lambda_1} - e^{iz})^2 (e^{i\lambda_2} - e^{iz})^2} d\lambda_1 d\lambda_2. \quad (2.5)$$

On the other hand, denoting

$$Q_n(z) = \int_{-\pi}^{\pi} \frac{e^{i\lambda} + e^{iz}}{e^{i\lambda} - e^{iz}} (V'(\lambda) - V'(\mu)) \rho_n(\lambda) d\lambda, \quad (2.6)$$

for $z = \mu + i\eta$, from (2.3) we will get

$$\mathcal{I}_n(z) = Q_n(z) + V'(\mu) f_n(z). \quad (2.7)$$

Finally from (2.5) and (2.7) we obtain the "square" equation

$$f_n^2(z) - 2iV'(\mu) f_n(z) - 2iQ_n(z) - 1 = -\frac{2}{n^2} G_n(z), \quad (2.8)$$

with

$$G_n(z) = \int |K_n(\lambda_1, \lambda_2)|^2 \frac{(e^{i\lambda_1} - e^{i\lambda_2})^2 e^{2iz}}{(e^{i\lambda_1} - e^{iz})^2 (e^{i\lambda_2} - e^{iz})^2} d\lambda_1 d\lambda_2$$

To proceed further we need some properties of the reproducing kernel K_n .

Lemma 2.1 Let $K_n(\lambda, \mu)$ be defined by (1.11). Then under the conditions of Theorem 1.2 for any $\delta > 0$

$$\left| \int (e^{i\lambda} - e^{i\mu}) |K_n(\lambda, \mu)|^2 d\mu \right| \leq \frac{1}{2} \left[\left| \psi_{n-1}^{(n)}(\lambda) \right|^2 + \left| \psi_n^{(n)}(\lambda) \right|^2 \right], \quad (2.9)$$

$$\int |e^{i\lambda} - e^{i\mu}|^2 |K_n(\lambda, \mu)|^2 d\mu \leq \left[\left| \psi_{n-1}^{(n)}(\lambda) \right|^2 + \left| \psi_n^{(n)}(\lambda) \right|^2 \right], \quad (2.10)$$

$$\int |e^{i\lambda} - e^{i\mu}|^2 |K_n(\lambda, \mu)|^2 d\lambda d\mu \leq 2, \quad (2.11)$$

$$\int_{|e^{i\lambda} - e^{i\mu}| > \delta} |K_n(\lambda, \mu)|^2 d\mu \leq \delta^{-2} \left[\left| \psi_{n-1}^{(n)}(\lambda) \right|^2 + \left| \psi_n^{(n)}(\lambda) \right|^2 \right], \quad (2.12)$$

$$\int_{|e^{i\lambda} - e^{i\mu}| > \delta} |K_n(\lambda, \mu)|^2 d\lambda d\mu \leq 2\delta^{-2}. \quad (2.13)$$

It is easy to see that $|e^{i\lambda} - e^{iz}| > C|\eta|$ if $|\eta| < 1$ for some $C > 0$. Hence, from (2.11) and (2.8), we derive

$$f_n^2(z) - 2iV'(\mu)f_n(z) - 2iQ_n(z) - 1 = O(n^{-2}\eta^{-4}). \quad (2.14)$$

Lemma 2.2 Under the conditions of Theorem 1.2 for any $d > 0$ and $\lambda \in [a + d, b - d]$

$$\rho_n(\lambda) \leq C, \quad (2.15)$$

$$\left| \frac{d\rho_n(\lambda)}{d\lambda} \right| \leq C_1 \left(\left| \psi_n^{(n)}(\lambda) \right|^2 + \left| \psi_{n-1}^{(n)}(\lambda) \right|^2 \right) + C_2. \quad (2.16)$$

From the conditions of Theorem 1.2, we obtain that $V''(\lambda)$ is bounded on the interval $[a, b]$. Hence we have for $\mu \in [a + d, b - d]$ and sufficiently small η

$$\begin{aligned} |Q_n(\mu + i\eta) - Q_n(\mu)| &\leq |e^{-\eta} - 1| \int_{-\pi}^{\pi} \frac{|V'(\lambda) - V'(\mu)| \rho_n(\mu)}{|e^{i\lambda} - e^{i\mu}| |e^{i\lambda} - e^{iz}|} d\lambda \leq \\ &\leq C\eta \left(\int_{|\lambda - \mu| < d/2} \frac{d\lambda}{|(\lambda - \mu)^2 + \eta^2|^{1/2}} + \int_{|\lambda - \mu| > d/2} \frac{\rho_n(\lambda) d\lambda}{|(\lambda - \mu)^2 + \eta^2|^{1/2} |\lambda - \mu|} \right) \leq \\ &\leq C\eta \ln^{-1} \eta + C\eta d^{-2} \leq C\eta \ln^{-1} \eta. \end{aligned} \quad (2.17)$$

Besides, applying (1.4), we get for $\phi(\lambda) = \frac{e^{i\lambda} + e^{i\mu}}{e^{i\lambda} - e^{i\mu}} (V'(\lambda) - V'(\mu))$

$$Q_n(\mu) = Q(\mu) + O(n^{-1/2} \ln^{1/2} n), \quad (2.18)$$

where

$$Q(\mu) = \int_{-\pi}^{\pi} \frac{e^{i\lambda} + e^{i\mu}}{e^{i\lambda} - e^{i\mu}} (V'(\lambda) - V'(\mu)) \rho(\lambda) d\lambda. \quad (2.19)$$

Combining (2.17) and (2.18), we find

$$Q_n(\mu + i\eta) = Q(\mu) + O(\eta \ln^{-1} \eta) + O(n^{-1/2} \ln^{1/2} n). \quad (2.20)$$

Now it is easy to see that we have from (2.20) and (2.8) for $z = \mu + in^{-4/9}$

$$f_n^2(z) - 2iV'(\mu)f_n(z) - 2iQ(\mu) - 1 = O(n^{-2/9}). \quad (2.21)$$

Lemma 2.3

$$\rho(\mu) = \frac{1}{2\pi} \sqrt{2iQ(\mu) + 1 - (V'(\mu))^2}. \quad (2.22)$$

Lemma 2.3 and equation (2.21) implies that for $z = \mu + in^{-4/9}$

$$\frac{1}{2\pi} \Re f_n(z) = \rho(\mu) + O(n^{-2/9}) \rho^{-1}(\mu). \quad (2.23)$$

Lemma 2.4 For $d > 0$, $k = n - 1, n$ and $\lambda \in [a + d, b - d]$

$$\int_{|\lambda - \mu| < n^{-1/4}} \left| \psi_k^{(n)}(\lambda) \right|^2 d\lambda \leq C n^{-1/4}, \quad (2.24)$$

$$\left| \psi_k^{(n)}(\lambda) \right|^2 \leq C n^{7/8}, \quad |\mu - \lambda| \leq n^{-1/4}. \quad (2.25)$$

Taking into account (2.23), to prove Theorem (1.4) it is enough to show that $\frac{1}{2\pi} \Re f_n(z) = \rho_n(\mu) + O(n^{-2/9})$. We use an evident relation

$$\Re \frac{e^{i\lambda} + e^{iz}}{e^{i\lambda} - e^{iz}} = \frac{\sinh \eta}{\cosh \eta - \cos(\lambda - \mu)} = \frac{d}{d\lambda} 2 \arctan \left(\tan \left(\frac{\lambda - \mu}{2} \right) \coth \left(\frac{\eta}{2} \right) \right).$$

Combining the relation $\frac{1}{2\pi} \int \Re \frac{e^{i\lambda} + e^{iz}}{e^{i\lambda} - e^{iz}} = 1$ with (2.15), we obtain

$$\begin{aligned} & \left| \frac{1}{2\pi} f_n(z) - \rho_n(\mu) \right| = \\ &= (2\pi)^{-1} \left| \left(\int_{|\mu - \lambda| \leq \eta^{1/2}} + \int_{\eta^{1/2} \leq |\mu - \lambda| \leq d/2} + \int_{|\mu - \lambda| \geq d/2} \right) \frac{\sinh \eta}{\cosh \eta - \cos(\lambda - \mu)} (\rho_n(\lambda) - \rho_n(\mu)) d\lambda \right| \leq \\ & \leq C \left| \int_{|s| \leq \eta^{1/2}} \frac{\sinh \eta}{\cosh \eta - \cos s} (\rho_n(s + \mu) - \rho_n(\mu)) d\lambda \right| + C\eta^{1/2} + C\eta. \end{aligned}$$

Using (2.16) and (2.24), we get finally

$$\left| \frac{1}{2\pi} f_n(z) - \rho_n(\mu) \right| \leq C \int_{|s| < \eta^{1/2}} |\rho'_n(s)| ds + C\eta^{1/2} \leq C\eta^{1/2}.$$

Theorem 1.4 is proved. \blacksquare

Now we pass to the proof of Theorem 1.2. We will use the following representation of K_n , which can be derived from the well-known identities of random matrix theory (see [1])

$$\begin{aligned} \frac{1}{n} K_n(\lambda, \mu) &= \frac{1}{n} \sum_{j=0}^{n-1} \psi_l^{(n)}(\lambda) \overline{\psi_l^{(n)}(\mu)} = \\ &= Q_{n,2}^{-1} e^{-n(V(\lambda) + V(\mu))/2} \int \prod_{j=2}^n (e^{i\lambda} - e^{i\lambda_j}) (e^{-i\mu} - e^{-i\lambda_j}) e^{-nV(\lambda_j)} d\lambda_j \prod_{2 \leq j < k \leq n} |e^{i\lambda_j} - e^{i\lambda_k}|^2, \end{aligned} \quad (2.26)$$

where $Q_{n,2} = n! \prod_{j=0}^{n-1} |\gamma_l^{(n)}|^{-2}$ and $\gamma_l^{(n)}$ is the coefficient in front of $e^{il\lambda}$ in the function $P_l^{(n)}$.

Remark 2.5 Consider the determinant (see (1.2))

$$\det \left\{ e^{ik\lambda_j} \right\}_{k,j=0}^{n-1} = e^{i(n-1) \sum \lambda_j / 2} \det \left\{ e^{i(k-(n-1)/2)\lambda_j} \right\}_{k,j=0}^{n-1}.$$

Taking the complex conjugate, we obtain

$$\overline{\det \{e^{ik\lambda_j}\}_{k,j=0}^{n-1}} = e^{-i(n-1)\sum \lambda_j/2} \det \left\{ e^{-i(k-(n-1)/2)\lambda_j} \right\}_{k,j=0}^{n-1} = (-1)^{[n/2]} e^{-i(n-1)\sum \lambda_j/2} \det \left\{ e^{i(k-(n-1)/2)\lambda_j} \right\}_{k,j=0}^{n-1}.$$

Thus, from (2.26) we get that the function $e^{-i(n-1)(\lambda-\mu)/2} K_n(\lambda, \mu)$ is real valued.

Now denote

$$\tilde{\mathcal{K}}_n(x, y) = \frac{1}{n} K_n\left(\lambda_0 + \frac{x}{n}, \lambda_0 + \frac{y}{n}\right), \quad \mathcal{K}_n(x, y) = e^{-i(n-1)(x-y)/2n} \tilde{\mathcal{K}}_n(x, y). \quad (2.27)$$

From the above we have that $\mathcal{K}_n(x, y)$ is a real valued and symmetric function. We get from (1.11), (1.12) and (1.13)

$$\int_{-n\pi}^{n\pi} \mathcal{K}_n(x, z) \mathcal{K}_n(z, y) dz = \mathcal{K}_n(x, y), \quad |\mathcal{K}_n(x, y)|^2 \leq \mathcal{K}_n(x, x) \mathcal{K}_n(y, y), \quad (2.28)$$

$$\mathcal{K}_n(x, x) = \rho_n(\lambda_0 + x/n) \leq C, \quad |\mathcal{K}_n(x, y)| \leq C, \quad \text{for } |x|, |y| \leq nd_0/2 \quad (2.29)$$

Differentiating in (2.26) $\tilde{\mathcal{K}}_n(x, y)$ with respect to x for $\lambda = \lambda_0 + x/n$, $\mu = \mu_0 + y/n$, we get

$$\begin{aligned} \frac{\partial}{\partial x} \tilde{\mathcal{K}}_n(x, y) &= -\frac{1}{2} V'(\lambda) \tilde{\mathcal{K}}_n(x, y) + \\ &+ \frac{n-1}{Q_{n,2}} e^{-n(V(\lambda)+V(\mu))/2} \int \frac{ie^{i\lambda}}{e^{i\lambda} - e^{i\lambda_2}} \prod_{j=2}^n (e^{i\lambda} - e^{i\lambda_j}) (e^{-i\mu} - e^{-i\lambda_j}) d\lambda_j \prod_{2 \leq j < k \leq n} |e^{i\lambda_j} - e^{i\lambda_k}|^2 = \\ &= -\frac{1}{2} V'(\lambda) \tilde{\mathcal{K}}_n(x, y) + \frac{i}{n^2} \int_{-\pi}^{\pi} \frac{e^{i\lambda}}{e^{i\lambda} - e^{i\lambda_2}} (K_n(\lambda_2, \lambda_2) K_n(\lambda, \mu) - K_n(\lambda, \lambda_2) K_n(\lambda_2, \mu)) d\lambda_2 = \\ &= -\frac{1}{2} V'(\lambda) \tilde{\mathcal{K}}_n(x, y) + \frac{i}{2n^2} \int_{-\pi}^{\pi} \frac{e^{i\lambda} + e^{i\lambda_2}}{e^{i\lambda} - e^{i\lambda_2}} (K_n(\lambda_2, \lambda_2) K_n(\lambda, \mu) - K_n(\lambda, \lambda_2) K_n(\lambda_2, \mu)) d\lambda_2 + \\ &\quad + \frac{i(n-1)}{2n^2} K_n(\lambda, \mu) = \\ &= -\frac{1}{2} V'(\lambda) \tilde{\mathcal{K}}_n(x, y) + \frac{1}{2n} \int_{-n\pi}^{n\pi} \cot\left(\frac{x-z}{2n}\right) (\tilde{\mathcal{K}}_n(z, z) \tilde{\mathcal{K}}_n(x, y) - \tilde{\mathcal{K}}_n(x, z) \tilde{\mathcal{K}}_n(z, y)) dz \\ &\quad + \frac{i(n-1)}{2n} \tilde{\mathcal{K}}_n(x, y) \quad (2.30) \end{aligned}$$

Lemma 2.6 Denote

$$D(\lambda) = V'(\lambda) + v.p. \int_{-\pi}^{\pi} \cot \frac{s}{2} \rho_n(\lambda + s) ds.$$

Then for any $d > 0$ we have uniformly in $[a+d, b-d]$

$$|D(\lambda)| \leq C n^{-1/4} \ln n.$$

Definition of \mathcal{K}_n (2.27), above Lemma, and the bound (2.29) yield

$$\frac{\partial}{\partial x} \mathcal{K}_n(x, y) = \frac{1}{2n} v.p. \int_{-n\pi}^{n\pi} \cot\left(\frac{z-x}{2n}\right) \mathcal{K}_n(x, z) \mathcal{K}_n(z, y) dz + O(n^{-1/4} \ln n). \quad (2.31)$$

Below we take $|x|, |y| \leq \mathcal{L} = \ln n$. Then from the inequality $|z| \leq n\pi$ we get $\left|\frac{x-z}{2n}\right| \leq 3\pi/4$. Function $x \cot x$ is bounded on $[0, 3\pi/4]$, thus

$$\left|\frac{1}{2n} \cot\left(\frac{x-z}{2n}\right)\right| \leq C \left|\frac{1}{x-z}\right|.$$

For $|x|, |y| \leq \mathcal{L}$ we can restrict integration in (2.31) by the domain $|z| \leq 2\mathcal{L}$, replacing $O(n^{-1/4} \ln n)$ by $O(\mathcal{L}^{-1})$. This follows from the bound

$$\left|\frac{1}{2n} \int_{2\mathcal{L} \leq |z| \leq n\pi} \cot\left(\frac{x-z}{2n}\right) \mathcal{K}_n(x, z) \mathcal{K}_n(z, y) dz\right| \leq C\mathcal{L}^{-1} \int |\mathcal{K}_n(x, z)| |\mathcal{K}_n(z, y)| dz \leq C\mathcal{L}^{-1}.$$

Note that

$$\frac{1}{2n} \cot \frac{x}{2n} - \frac{1}{x} = O(n^{-2} \ln n), \quad \text{for } x = O(\ln n)$$

Hence, we get from the above estimates and (2.31)

$$\frac{\partial}{\partial x} \mathcal{K}_n(x, y) = v.p. \int_{|z| \leq 2\mathcal{L}} \frac{\mathcal{K}_n(x, z) \mathcal{K}_n(z, y)}{z-x} dz + O(\mathcal{L}^{-1}). \quad (2.32)$$

Next lemma shows that \mathcal{K}_n behave almost like a difference kernel.

Lemma 2.7 For any $d > 0$ we have uniformly in $\lambda_0 \in [a+d, b-d]$ and $|x|, |y| \leq nd/4$

$$\left|\frac{\partial}{\partial x} \mathcal{K}_n(x, y) + \frac{\partial}{\partial y} \mathcal{K}_n(x, y)\right| \leq C \left(n^{-1/8} + |x-y|n^{-2}\right), \quad (2.33)$$

$$|\mathcal{K}_n(x, y) - \mathcal{K}_n(0, y-x)| \leq C|x| \left(n^{-1/8} + |x-y|n^{-2}\right). \quad (2.34)$$

Remark 2.8 Note that the last inequality with $\lambda_0 + x_1/n$ instead of λ_0 , and $x_2 - x_1$ instead x and y , leads to the bound, valid for any $|x_{1,2}| \leq nd_0/8$

$$|\mathcal{K}_n(x_2, x_2) - \mathcal{K}_n(x_1, x_1)| \leq Cn^{-1/8} |x_2 - x_1|. \quad (2.35)$$

Lemma 2.9 For any $|x|, |y| \leq \mathcal{L}$

$$\left|\frac{\partial}{\partial x} \mathcal{K}_n(x, y)\right| \leq C, \quad \int_{|x| \leq \mathcal{L}} \left|\frac{\partial}{\partial x} \mathcal{K}_n(x, y)\right|^2 dx \leq C. \quad (2.36)$$

Denote

$$\begin{aligned} \mathcal{K}_n^*(x) &= \mathcal{K}_n(x, 0) \mathbf{1}_{|x| \leq \mathcal{L}} + \mathcal{K}_n(\mathcal{L}, 0)(1 + \mathcal{L} - x) \mathbf{1}_{\mathcal{L} < x \leq \mathcal{L}+1} \\ &+ \mathcal{K}_n(-\mathcal{L}, 0)(1 + \mathcal{L} + x) \mathbf{1}_{-\mathcal{L}-1 \leq x < -\mathcal{L}}, \end{aligned} \quad (2.37)$$

and observe that for $y = 0$ and for any $|x| \leq \mathcal{L}/3$ similarly to (2.32) we can restrict integration in (2.32) to $|z| \leq 2\mathcal{L}/3$, with a mistake $O(\mathcal{L}^{-1})$. This and Lemma 2.7 give us the equation

$$\frac{\partial}{\partial x} \mathcal{K}_n^*(x) = \int_{|z| \leq 2\mathcal{L}/3} \frac{\mathcal{K}_n^*(z) \mathcal{K}_n^*(x-z)}{z} dz + r_n(x) + O(\mathcal{L}^{-1}), \quad (2.38)$$

where

$$r_n(x) = \int_{|z| \leq 2\mathcal{L}/3} \frac{\mathcal{K}_n(z, 0)(\mathcal{K}_n(x, z) - \mathcal{K}_n(0, x - z))}{z} dz,$$

and by Lemma 2.7 we have for $|x| \leq \mathcal{L}/3$

$$r_n(y) = O(n^{-1/8} \log n).$$

Now, using the estimates similiar to (2.32), we can restrict integration in (2.38) to the real axis. From Lemma 2.9 and relations (2.28), (2.29) we get

$$\int |\mathcal{K}_n^*(x)|^2 dx \leq \int |\mathcal{K}_n(x, 0)|^2 dx + C' \leq C, \quad \int \left| \frac{d}{dx} \mathcal{K}_n^*(x) \right|^2 dx \leq C. \quad (2.39)$$

Consider the Fourier transform

$$\widehat{\mathcal{K}}_n^*(p) = \int \mathcal{K}_n^*(x) e^{ipx} dx,$$

where the integral is defined in the $L^2(\mathbb{R})$ sense, and write $\mathcal{K}_n^*(x)$ as

$$\mathcal{K}_n^*(x) = (2\pi)^{-1} \int \widehat{\mathcal{K}}_n^*(p) e^{-ipx} dp. \quad (2.40)$$

From (1.19) we have

$$\int \widehat{\mathcal{K}}_n^*(p) dp = 2\pi \rho(\lambda_0) + o(1), \quad (2.41)$$

and from (2.39) and the Parseval equation

$$\int p^2 |\widehat{\mathcal{K}}_n^*(p)|^2 dp \leq C. \quad (2.42)$$

From the definition of $\mathcal{K}_n(x, y)$ we get that the kernel is positive definite

$$\int_{-\mathcal{L}}^{\mathcal{L}} \mathcal{K}_n(x, y) f(x) \overline{f}(y) dx dy \geq 0, \quad f \in L_2(\mathbb{R}),$$

therefore from (2.34) we have for any function $f \in L_2(\mathbb{R})$:

$$\int \widehat{\mathcal{K}}_n^*(p) |\widehat{f}(p)|^2 dp \geq -C \|f\|_{L^2(\mathbb{R})}^2 (n^{-1/8} \log^4 n + O(\mathcal{L}^{-1})). \quad (2.43)$$

It is easy to see that it follows from the Parseval equation and (2.34) that

$$\int |\widehat{\mathcal{K}}_n^*(p) - \widehat{\mathcal{K}}_n^*(-p)|^2 dp \leq 2\pi \int |\mathcal{K}_n^*(x) - \mathcal{K}_n^*(-x)|^2 dx \leq C n^{-1/8} \log^3 n. \quad (2.44)$$

By the definition of singular integrals

$$\int \frac{\mathcal{K}_n^*(z) \mathcal{K}_n^*(x - z)}{z} dz = \lim_{\varepsilon \rightarrow +0} \int dz \mathcal{K}_n^*(z) \mathcal{K}_n^*(y - z) \Re(z + i\varepsilon)^{-1}. \quad (2.45)$$

In accordance with the relation

$$\int e^{ipz} \Re(z + i\varepsilon)^{-1} dz = \pi i e^{-\varepsilon|p|} \operatorname{sgn} p$$

and the Parseval equation we can write the r.h.s. of (2.38) as

$$\begin{aligned} \frac{i}{4\pi} \lim_{\varepsilon \rightarrow +0} \int dp dp' \widehat{\mathcal{K}}_n^*(p) \widehat{\mathcal{K}}_n^*(p') e^{-ipx} \text{sign}(p-p') e^{-\varepsilon|p-p'|} \\ = \frac{i}{2\pi} \int dp e^{-ipx} \widehat{\mathcal{K}}_n^*(p) \int_0^p \widehat{\mathcal{K}}_n^*(p') dp' \\ - \frac{i}{4\pi} \int dp e^{-ipx} \widehat{\mathcal{K}}_n^*(p) \int_0^\infty (\widehat{\mathcal{K}}_n^*(p') - \widehat{\mathcal{K}}_n^*(-p')) dp'. \end{aligned} \quad (2.46)$$

Note that both integrals are absolutely convergent because $\widehat{\mathcal{K}}_n^* \in L^1(\mathbb{R})$ by (2.42). Now using the Schwarz inequality and (2.42), we can estimate the second component

$$\begin{aligned} \left| \int_0^\infty (\widehat{\mathcal{K}}_n^*(p') - \widehat{\mathcal{K}}_n^*(-p')) dp' \right| &\leq \left| \int_0^{\mathcal{L}^2} (\widehat{\mathcal{K}}_n^*(p') - \widehat{\mathcal{K}}_n^*(-p')) dp' \right| \\ &+ \int_{|p| > \mathcal{L}^2} |\widehat{\mathcal{K}}_n^*(p')| dp' \leq \mathcal{L} \left(\int |\widehat{\mathcal{K}}_n^*(p') - \widehat{\mathcal{K}}_n^*(-p')|^2 dp' \right)^{1/2} + C\mathcal{L}^{-1}. \end{aligned}$$

Thus, from (2.44) - (2.46) we have uniformly in $|x| < \mathcal{L}/3$

$$\int \frac{\mathcal{K}_n^*(z) \mathcal{K}_n^*(x-z)}{z} dz = \frac{i}{2\pi} \int dp \widehat{\mathcal{K}}_n^*(p) e^{-ipx} \int_0^p \widehat{\mathcal{K}}_n^*(p') dp' + O(\mathcal{L}^{-1}).$$

This allows us to transform (2.38) into the following asymptotic relation, valid for $|x| \leq \mathcal{L}/3$:

$$\int \widehat{\mathcal{K}}_n^*(p) \left(\int_0^p \widehat{\mathcal{K}}_n^*(p') dp' - p \right) e^{-ipx} dp = O(\mathcal{L}^{-1}). \quad (2.47)$$

Consider the functions

$$F_n(p) = \int_0^p \widehat{\mathcal{K}}_n^*(p') dp'. \quad (2.48)$$

Since $p\widehat{\mathcal{K}}_n^*(p) \in L^2(\mathbb{R})$, the sequence $\{F_n(p)\}$ consists of functions that are uniformly bounded and equicontinuous on \mathbb{R} . Thus $\{F_n(p)\}$ is a compact family with respect to uniform convergence. Hence, the limit F of any subsequence $\{F_{n_k}\}$ possesses the properties:

- (a) F is bounded and continuous;
- (b) $F(p) = -F(-p)$ (see (2.44));
- (c) $F(p) \leq F(p')$, if $p \leq p'$ (see (2.43));
- (d) $F(+\infty) - F(-\infty) = 2\pi\rho(\lambda_0)$ (see (2.41));
- (e) the following equation is valid for any smooth function g with the compact support (see (2.47)):

$$\int (F(p) - p)g(p)dF(p) = 0. \quad (2.49)$$

The last property implies that $F(p) = p$ or $F(p) = \text{const}$, hence it follows from (a) - (c) that

$$F(p) = p \mathbf{1}_{|p| \leq p_0} + p_0 \text{sign}(p) \mathbf{1}_{|p| \geq p_0},$$

where $p_0 = \pi\rho(\lambda_0)$ from (d). We conclude that (2.49) is uniquely solvable, thus the sequence $\{F_n\}$ converges uniformly on any compact to the above F . This and (2.48) imply the weak convergence of the sequence $\{\mathcal{K}_n^*\}$ to the function $\rho(\lambda_0)S(\rho(\lambda_0)x)$, where $S(x)$ is defined in (1.8). But weak convergence combined with (2.29) and (2.36) implies the uniform convergence of $\{\mathcal{K}_n^*\}$ to \mathcal{K}^* on any interval. Thus we have uniformly in (x, y) , varying on a compact set of \mathbb{R}^2

$$\lim_{n \rightarrow \infty} \mathcal{K}_n(x, y) = \rho(\lambda_0)S(\rho(\lambda_0)(x - y)).$$

Recalling all definitions, we conclude that Theorem 1.2 is proved.

Auxiliary results for Theorem 1.2

Proof of lemma 2.1. Denote

$$r_{k,j}^{(n)} = \int_{-\pi}^{\pi} e^{i\lambda} \psi_k^{(n)}(\lambda) \overline{\psi_j^{(n)}(\lambda)} d\lambda. \quad (2.50)$$

Note that from the orthogonality (2.66) we have $r_{k,j}^{(n)} = 0$ for $j > k + 1$. Thus,

$$e^{i\lambda} \psi_k^{(n)}(\lambda) = \sum_{j=0}^{k+1} r_{k,j}^{(n)} \psi_j^{(n)}(\lambda). \quad (2.51)$$

Multiplication on $e^{i\lambda}$ is isometric in $L_2[-\pi, \pi]$, therefore

$$\sum_{j=0}^{k+1} |r_{k,j}^{(n)}|^2 = \left\| \psi_k^{(n)}(\lambda) \right\|_2^2 = 1.$$

Now we are ready to prove (2.9)

$$\begin{aligned} \int_{-\pi}^{\pi} \left(e^{i\lambda} - e^{i\mu} \right) |K_n(\lambda, \mu)|^2 d\mu &= \\ &= e^{i\lambda} K_n(\lambda, \lambda) - \int_{-\pi}^{\pi} e^{i\mu} \sum_{m=0}^{n-1} \psi_m^{(n)}(\mu) \overline{\psi_m^{(n)}(\lambda)} \sum_{l=0}^{n-1} \psi_l^{(n)}(\lambda) \overline{\psi_l^{(n)}(\mu)} d\mu = \\ &= e^{i\lambda} K_n(\lambda, \lambda) - \sum_{l,m=0}^{n-1} r_{m,l}^{(n)} \psi_l^{(n)}(\lambda) \overline{\psi_m^{(n)}(\lambda)} = \\ &= r_{n-1,n}^{(n)} \psi_{n-1}^{(n)}(\lambda) \overline{\psi_n^{(n)}(\lambda)}. \end{aligned} \quad (2.52)$$

Now, using the Cauchy inequality and the bound $|r_{n-1,n}^{(n)}| \leq 1$, we get (2.9). Similarly, it is easy to obtain the relation

$$\int_{-\pi}^{\pi} \left| e^{i\lambda} - e^{i\mu} \right|^2 |K_n(\lambda, \mu)|^2 d\mu = 2\Re \left\{ e^{i\lambda} r_{n-1,n}^{(n)} \overline{\psi_{n-1}^{(n)}(\lambda)} \psi_n^{(n)}(\lambda) \right\},$$

which implies (2.10). Bounds (2.11), (2.12), (2.13) are evident consequences of (2.10). Lemma is proved. \blacksquare

Proof of Lemma 2.2. Observe that

$$\frac{d\rho_n(\lambda)}{d\lambda} = \frac{d\rho_n(\lambda+t)}{dt} \Big|_{t=0}.$$

Changing variables in (1.5) $\lambda_j = \mu_j + t$, in view of periodicity of all functions in consideration, we have the representation for $\rho_n(\lambda + t)$

$$\rho_n(\lambda + t) = \frac{1}{Z_n} \int e^{-nV(\lambda+t)} \prod_{2 \leq j < k \leq n} |e^{i\mu_j} - e^{i\mu_k}|^2 \prod_{j=2}^n e^{-nV(\mu_j+t)} |e^{i\lambda} - e^{i\mu_j}|^2 d\mu_j.$$

After differentiating with respect to t , for $t = 0$ we get

$$\begin{aligned} \frac{d\rho_n(\lambda)}{d\lambda} &= -nV'(\lambda)p_1^{(n)}(\lambda) - n(n-1) \int_{-\pi}^{\pi} V'(\mu)p_2^{(n)}(\lambda, \mu) d\mu = \\ &= -V'(\lambda)K_n(\lambda, \lambda) - \int_{-\pi}^{\pi} V'(\mu) \left[K_n(\lambda, \lambda)K_n(\mu, \mu) - |K_n(\lambda, \mu)|^2 \right] d\mu. \end{aligned} \quad (2.53)$$

Since $V'(\lambda)$ is an odd function, and $K_n(\lambda, \lambda)$ is an even function, we obtain

$$\int_{-\pi}^{\pi} V'(\lambda)K_n(\lambda, \lambda) d\lambda = 0.$$

Thus, from (2.53) we get

$$\rho'_n(\lambda) = \int_{-\pi}^{\pi} (V'(\mu) - V'(\lambda)) |K_n(\lambda, \mu)|^2 d\mu. \quad (2.54)$$

We split this integral in two parts corresponding to the domains $|\mu - \lambda| \leq d/2$ and $|\mu - \lambda| \geq d/2$. In the second integral we will use (2.12). It follows from (1.17) that in the first integral we can rewrite $V'(\lambda)$ as

$$\begin{aligned} V'(\mu) - V'(\lambda) &= (\mu - \lambda)V''(\lambda) + O(|\mu - \lambda|^2) = \\ &= (e^{i\mu} - e^{i\lambda}) \frac{V''(\lambda)}{ie^{i\lambda}} + O\left((e^{i\mu} - e^{i\lambda})^2\right), \end{aligned}$$

and using (2.9) and (2.10), we get (2.16). To prove (2.15) we use the next well-known inequality

Proposition 2.10 For any function $u : [a_1, b_1] \rightarrow \mathbb{C}$ with $u' \in L_1(a_1, b_1)$ we have

$$\|u\|_{\infty} \leq \|u'\|_1 + (b_1 - a_1)^{-1} \|u\|_1, \quad (2.55)$$

where $\|\cdot\|_1, \|\cdot\|_{\infty}$ are the L_1 and uniform norms on the interval $[a_1, b_1]$.

This inequality can be obtained easily from the relation

$$u(\lambda) = \frac{1}{b_1 - a_1} \int_{a_1}^{b_1} (u(\lambda) - u(\mu)) d\mu + \frac{1}{b_1 - a_1} \int_{a_1}^{b_1} u(\mu) d\mu.$$

Using (2.55) for $u = \rho_n$ and interval $[a + d, b - d]$, we get (2.15). ■

Proof of Lemma 2.3. From (1.4) and (2.21) we have for non-real z

$$f^2(z) - 2iV'(\mu)f(z) - 2iQ(z) - 1 = 0, \quad (2.56)$$

where $f(z)$ is the Gergmglotz transformation of the limiting density $\rho(\lambda)$. From (2.19) and (2.2), it is easy to see that $Q(\mu + i0)$ is an imaginary valued, bounded, continuous function. And from (2.2) we obtain

$$\rho(\mu) = \frac{1}{2\pi} \Re f(\mu + i0).$$

Computing imaginary and real parts in (2.56), we get the relations

$$\Im f(\mu + i0) = V'(\mu), \quad (2.57)$$

$$\Re f(\mu + i0) = \sqrt{2iQ(\mu) + 1 - (V'(\mu))^2}, \quad (2.58)$$

from which we obtain (2.22). ■

Proof of Lemma 2.4. To prove (2.24) with $k = n - 1$ we introduce the probability density

$$p_n^-(\lambda_1, \dots, \lambda_{n-1}) = \frac{1}{Z_n^-} \prod_{1 \leq j < k \leq n-1} \left| e^{i\lambda_j} - e^{i\lambda_k} \right|^2 \exp \left\{ -n \sum_{j=1}^{n-1} V(\lambda_j) \right\}. \quad (2.59)$$

Denote

$$\rho_n^-(\lambda) = \frac{n-1}{n} \int p_n^-(\lambda, \lambda_2, \dots, \lambda_{n-1}) d\lambda_2 \dots d\lambda_{n-1} = \frac{1}{n} \sum_{j=0}^{n-2} \left| \psi_j^{(n)}(\lambda) \right|^2. \quad (2.60)$$

Thus we get

$$\left| \psi_{n-1}^{(n)}(\lambda) \right|^2 = n(\rho_n(\lambda) - \rho_n^-(\lambda)). \quad (2.61)$$

Analogously to equation (2.8) we can obtain the "square" equation

$$\frac{i}{2} [f_n^-(z)]^2 + \int_{-\pi}^{\pi} \frac{e^{i\lambda} + e^{iz}}{e^{i\lambda} - e^{iz}} V'(\lambda) \rho_n^-(\lambda) d\lambda = \frac{i}{2} + O(n^{-2}\eta^{-4}), \quad (2.62)$$

for the Germglotz transformation $f_n^-(z)$ of the function $\rho_n^-(\lambda)$. Denote

$$\Delta_n(z) = n(f_n(z) - f_n^-(z)) = \int_{-\pi}^{\pi} \frac{e^{i\lambda} + e^{iz}}{e^{i\lambda} - e^{iz}} \left| \psi_{n-1}^{(n)}(\lambda) \right|^2 d\lambda. \quad (2.63)$$

Subtracting (2.62) from (2.8) we obtain for $z = \mu + in^{-1/4}$

$$\begin{aligned} \frac{i}{2} \Delta_n(z) (f_n(z) + f_n^-(z)) &= - \int_{-\pi}^{\pi} \frac{e^{i\lambda} + e^{iz}}{e^{i\lambda} - e^{iz}} V'(\lambda) \left| \psi_{n-1}^{(n)}(\lambda) \right|^2 d\lambda + O(1), \\ \frac{i}{2} \Delta_n(z) (f_n(z) + f_n^-(z)) - 2iV'(\mu) &= \\ &= \int_{-\pi}^{\pi} \frac{e^{i\lambda} + e^{iz}}{e^{i\lambda} - e^{iz}} (V'(\mu) - V'(\lambda)) \left| \psi_{n-1}^{(n)}(\lambda) \right|^2 d\lambda + O(1) = O(1). \end{aligned}$$

Note that $\Re f_n^-(z) > 0$ for $\Im z > 0$ therefore

$$\Re \Delta_n(\mu + in^{-1/4}) \leq \frac{C}{\Re f_n(\mu + in^{-1/4})}$$

Analogously to (2.23) we can obtain for $z = \mu + in^{-1/4}$

$$\frac{1}{2\pi} \Re f_n(z) = \rho(\mu) + O(n^{-1/8}) \rho^{-1}(\mu),$$

hence $\Re f_n(z) \geq C_2$ for sufficiently big n , where C_2 is defined in (1.17). Thus

$$\Re \Delta_n \left(\mu + in^{-1/4} \right) \leq C.$$

Note that

$$\Re \frac{e^{i\lambda} + e^{iz}}{e^{i\lambda} - e^{iz}} = \frac{\sinh \eta}{\cosh \eta - \cos(\mu - \lambda)} \geq C \frac{\eta}{\eta^2 + (\mu - \lambda)^2},$$

for $\eta^2 + (\mu - \lambda)^2 < 1$. Thus

$$\begin{aligned} \int_{|\lambda - \mu| < n^{-1/4}} \left| \psi_{n-1}^{(n)}(\lambda) \right|^2 d\lambda &\leq 2n^{-1/2} \int_{|\lambda - \mu| < n^{-1/4}} \frac{\left| \psi_{n-1}^{(n)}(\lambda) \right|^2}{n^{-1/2} + (\mu - \lambda)^2} d\lambda \leq \\ &\leq Cn^{-1/4} \Re \Delta_n \left(\mu + in^{-1/4} \right) \leq Cn^{-1/4}. \end{aligned}$$

A similar bound can be obtained for $\psi_n^{(n)}(\lambda)$ by using the densities:

$$\begin{aligned} p_n^+(\lambda_1, \dots, \lambda_{n+1}) &= \frac{1}{Q_{n,2}^+} \prod_{1 \leq j \leq n+1} e^{-nV(\lambda_j)} \prod_{1 \leq j < k \leq n+1} \left| e^{i\lambda_j} - e^{i\lambda_k} \right|^2, \\ \rho_n^+(\lambda) &= \frac{n+1}{n} \int p_n^+(\lambda, \lambda_2, \dots, \lambda_{n+1}) d\lambda_2 \dots d\lambda_{n+1} = \frac{1}{n} \sum_{j=0}^n \left| \psi_j^{(n)}(\lambda) \right|^2. \end{aligned}$$

Analogously we will have $\left| \psi_n^{(n)}(\lambda) \right|^2 = n(\rho_n^+(\lambda) - \rho_n(\lambda))$. Thus, the estimate (2.24) is proved. Now we proceed to prove of (2.25) for $k = n$. We use the inequality

Proposition 2.11 For any C^1 function $u : [a_1, b_1] \rightarrow \mathbb{C}$

$$\|u\|_\infty^2 \leq 2 \|u\|_2 \|u'\|_2 + (b_1 - a_1)^{-1} \|u\|_2^2, \quad (2.64)$$

where $\|\cdot\|_2, \|\cdot\|_\infty$ are the L_2 and uniform norms on the interval $[a_1, b_1]$.

This inequality is a simple consequence of the relation

$$u^2(\lambda) = \frac{1}{b_1 - a_1} \int_{a_1}^{b_1} (u^2(\lambda) - u^2(\mu)) d\mu + \frac{1}{b_1 - a_1} \int_{a_1}^{b_1} u^2(\mu) d\mu$$

Consider the interval $\Delta = [\lambda - n^{-1/4}, \lambda + n^{-1/4}]$ and the function $\psi(\lambda) = \psi_n^{(n)}(\lambda)$. From the inequality we have

$$|\psi(\lambda)|^2 \leq 2 \|\psi\|_{2,\Delta} \|\psi'\|_{2,\Delta} + \frac{1}{2} n^{1/4} \|\psi\|_{2,\Delta}, \quad (2.65)$$

where $\|\cdot\|_{2,\Delta}$ is L_2 norm on the interval Δ . It is easy to see that

$$\|\psi\|_{2,\Delta} \leq \|\psi\|_{2,[-\pi,\pi]} = 1.$$

Denote $P(\lambda) = P_n^{(n)}(\lambda)$ and $\omega(\lambda) = e^{-nV(\lambda)/2}$, than $\psi(\lambda) = P(\lambda)\omega(\lambda)$. Now we estimate $\|\psi'\|_{2,[-\pi,\pi]}$:

$$\begin{aligned} \|\psi'\|_{2,[-\pi,\pi]} &= \|P'\omega + P\omega'\|_{2,[-\pi,\pi]} \leq \|P'\omega\|_{2,[-\pi,\pi]} + \|P\omega'\|_{2,[-\pi,\pi]} \\ \|P\omega'\|_{2,[-\pi,\pi]} &= \frac{n}{2} \|PV'\omega\|_{2,[-\pi,\pi]} \leq Cn \|P\omega\|_{2,[-\pi,\pi]} = Cn \end{aligned}$$

$$\begin{aligned} \|P'\omega\|_{2,[-\pi,\pi]}^2 &= \int P'(\lambda) \overline{P'(\lambda)} \omega^2(\lambda) d\lambda = - \int P(\lambda) \overline{P''(\lambda)} \omega^2(\lambda) d\lambda + \\ &\quad + n \int P(\lambda) \overline{P'(\lambda)} V'(\lambda) \omega^2(\lambda) d\lambda \end{aligned}$$

Using the orthogonality

$$\int e^{-im\lambda} \omega(\lambda) \psi_k^{(n)} d\lambda = 0, \quad \text{for } m < k, \quad (2.66)$$

we obtain

$$\int P(\lambda) \overline{P''(\lambda)} \omega^2(\lambda) d\lambda = \int P(\lambda) \gamma_n^{(n)} (-in)^2 e^{-in\lambda} \omega^2(\lambda) d\lambda = -in \int P(\lambda) \overline{P'(\lambda)} \omega^2(\lambda) d\lambda,$$

where $\gamma_n^{(n)}$ is defined in (2.26). Thus,

$$\|P'\omega\|_{2,[-\pi,\pi]}^2 = n \int P(\lambda) \overline{P'(\lambda)} (V'(\lambda) + i) \omega^2(\lambda) d\lambda \leq Cn \|P'\omega\|_{2,[-\pi,\pi]},$$

and we obtain that $\|P'\omega\|_{2,[-\pi,\pi]} \leq Cn$. Combining all above bounds, we conclude that $\|\psi'\|_{2,[-\pi,\pi]} \leq Cn$. Now using (2.65) and (2.24), we obtain (2.25) for $k = n$. For $k = n - 1$ the proof is the same. ■

Proof of Lemma 2.6. Similarly to (2.21) for $\eta = n^{-3/8}$ and $\mu \in [a + d, b - d]$ for f_n , defined in (2.3), we obtain

$$|\Im f_n(\mu + i\eta) - V'(\mu)| \leq Cn^{-3/8} \ln n. \quad (2.67)$$

Moreover, we estimate $M = \Im f_n(\mu + i\eta) + v.p. \int_{-\pi}^{\pi} \cot \frac{s}{2} \rho_n(\mu + s) ds$. Note that

$$\Im \frac{e^{i\lambda} + e^{iz}}{e^{i\lambda} - e^{iz}} = - \frac{\sin(\lambda - \mu)}{\cosh \eta - \cos(\lambda - \mu)}.$$

Hence,

$$\begin{aligned} M &= v.p. \int \left(\cot \frac{s}{2} - \frac{\sin s}{\cosh \eta - \cos s} \right) \rho_n(\mu + s) ds = \\ &= \int_{|s| \leq d/2} \ln \left(\frac{\cosh \eta - \cos s}{1 - \cos s} \right) \rho'_n(\mu + s) ds + O(\eta) = I_1 + I_2 + I_3 + O(\eta), \end{aligned}$$

where I_1 is the integral over $|s| \leq n^{-2}$, I_2 is the integral over $n^{-2} \leq |s| \leq n^{-1/4}$ and I_3 is the integral over $n^{-1/4} \leq |s| \leq d/2$. We estimate every term:

$$|I_1| \stackrel{(2.25)}{\leq} Cn^{7/8} \int_{|s| \leq n^{-2}} \ln \left(\frac{\cosh \eta - \cos s}{1 - \cos s} \right) ds \leq Cn^{-9/8} \ln n,$$

$$|I_2| \leq C \ln n \int_{n^{-2} \leq |s| \leq n^{1/4}} |\rho'_n(\mu + s)| ds \stackrel{(2.24)}{\leq} Cn^{-1/4} \ln n,$$

$$|I_3| \stackrel{(2.16)}{\leq} Cn^{-1/4} \int_{|s| \leq d/2} \left(\left| \psi_n^{(n)}(\mu + s) \right|^2 + \left| \psi_{n-1}^{(n)}(\mu + s) \right|^2 \right) ds \leq Cn^{-1/4}.$$

Combining above bounds with (2.67), we obtain that the lemma is proved. ■

Proof of Lemma 2.7. To simplify notations we denote for $t \in [0, 1]$

$$\lambda_x = \lambda_0 + \frac{x - tx}{n}, \quad \lambda_y = \lambda_0 + \frac{y - tx}{n}. \quad (2.68)$$

Then similarly to (2.30) and (2.54) we obtain

$$\frac{d}{dt} K_n(\lambda_x, \lambda_y) = x \int_{-\pi + \lambda_0}^{\pi + \lambda_0} K_n(\lambda_x, \lambda) K_n(\lambda, \lambda_y) \left(\frac{1}{2} V'(\lambda_x) + \frac{1}{2} V'(\lambda_y) - V'(\lambda) \right) d\lambda. \quad (2.69)$$

To get our estimates, we split this integral in two parts $|\lambda - \lambda_0| \leq d/2$ and $|\lambda - \lambda_0| \geq d/2$. By the assumption of the lemma λ_x, λ_y are in $[a + d/2, b - d/2]$, thus in the first integral we can write

$$\begin{aligned} V'(\lambda) - \frac{1}{2} V'(\lambda_x) - \frac{1}{2} V'(\lambda_y) &= \\ &= \left(e^{i\lambda} - e^{i\lambda_x} \right) \frac{V''(\lambda_x)}{2ie^{i\lambda_x}} + \left(e^{i\lambda} - e^{i\lambda_y} \right) \frac{V''(\lambda_y)}{2ie^{i\lambda_y}} + O \left(\left| e^{i\lambda} - e^{i\lambda_x} \right|^2 + \left| e^{i\lambda} - e^{i\lambda_y} \right|^2 \right) = \\ &= \left(e^{i\lambda} - e^{i\lambda_x} \right) \frac{V''(\lambda_x)}{2ie^{i\lambda_x}} + \left(e^{i\lambda} - e^{i\lambda_y} \right) \frac{V''(\lambda_y)}{2ie^{i\lambda_y}} + O \left(\left| e^{i\lambda} - e^{i\lambda_x} \right| \left| e^{i\lambda} - e^{i\lambda_y} \right| + \frac{|x - y|^2}{n^2} \right). \end{aligned}$$

Similarly to (2.52), we obtain

$$\int_{-\pi}^{\pi} K_n(\lambda_x, \lambda) K_n(\lambda, \lambda_y) \left(e^{i\lambda} - e^{i\lambda_x} \right) d\lambda = -r_{n-1,n}^{(n)} \psi_n^{(n)}(\lambda_x) \overline{\psi_{n-1}^{(n)}(\lambda_y)}.$$

Hence,

$$\int_{|\lambda - \lambda_0| \leq d/2} K_n(\lambda_x, \lambda) K_n(\lambda, \lambda_y) \left(e^{i\lambda} - e^{i\lambda_x} \right) d\lambda = -r_{n-1,n}^{(n)} \psi_n^{(n)}(\lambda_x) \overline{\psi_{n-1}^{(n)}(\lambda_y)} - I_d,$$

where

$$\begin{aligned} |I_d| &= \left| \int_{|\lambda - \lambda_0| \geq d/2} K_n(\lambda_x, \lambda) K_n(\lambda, \lambda_y) \left(e^{i\lambda} - e^{i\lambda_x} \right) d\lambda \right| \leq \\ &\leq C \left[\int_{|\lambda - \lambda_0| \geq d/2} |K_n(\lambda_x, \lambda)|^2 d\lambda \int_{|\lambda - \lambda_0| \geq d/2} |K_n(\lambda, \lambda_y)|^2 d\lambda \right]^{1/2} \stackrel{(2.12)}{\leq} \\ &\leq C \left[\left| \psi_{n-1}^{(n)}(\lambda_x) \right|^2 + \left| \psi_n^{(n)}(\lambda_x) \right|^2 + \left| \psi_{n-1}^{(n)}(\lambda_y) \right|^2 + \left| \psi_n^{(n)}(\lambda_y) \right|^2 \right]. \end{aligned}$$

The same bounds are valid for the term with the $e^{i\lambda_y}$ instead $e^{i\lambda_x}$. To estimate other terms we use the Schwarz inequality

$$\begin{aligned} \int_{|\lambda - \lambda_0| \leq d/2} \left| K_n(\lambda_x, \lambda) K_n(\lambda, \lambda_y) \left(e^{i\lambda} - e^{i\lambda_x} \right) \left(e^{i\lambda} - e^{i\lambda_y} \right) \right| d\lambda &\leq \\ &\leq \left[\int_{-\pi}^{\pi} \left| K_n(\lambda_x, \lambda) \left(e^{i\lambda} - e^{i\lambda_x} \right) \right|^2 d\lambda \int_{-\pi}^{\pi} \left| K_n(\lambda, \lambda_y) \left(e^{i\lambda} - e^{i\lambda_y} \right) \right|^2 d\lambda \right]^{1/2} \stackrel{(2.11)}{\leq} \\ &\leq C \left[\left| \psi_{n-1}^{(n)}(\lambda_x) \right|^2 + \left| \psi_n^{(n)}(\lambda_x) \right|^2 + \left| \psi_{n-1}^{(n)}(\lambda_y) \right|^2 + \left| \psi_n^{(n)}(\lambda_y) \right|^2 \right], \end{aligned}$$

$$\int_{|\lambda-\lambda_0|\leq d/2} |K_n(\lambda_x, \lambda) K_n(\lambda, \lambda_y)| d\lambda \leq n(\rho_n(\lambda_x) + \rho_n(\lambda_y)) \leq Cn$$

In the second integral we use that $V'(\lambda)$ is bounded, Cauchy inequality $|K_n(\lambda_x, \lambda) K_n(\lambda, \lambda_y)| \leq |K_n(\lambda_x, \lambda)|^2 + |K_n(\lambda, \lambda_y)|^2$ and the bound (2.12). Thus,

$$\left| \frac{d}{dt} K_n(\lambda_x, \lambda_y) \right| \leq C|x| \left[\left| \psi_{n-1}^{(n)}(\lambda_x) \right|^2 + \left| \psi_n^{(n)}(\lambda_x) \right|^2 + \left| \psi_{n-1}^{(n)}(\lambda_y) \right|^2 + \left| \psi_n^{(n)}(\lambda_y) \right|^2 + \frac{|x-y|}{n} \right] \quad (2.70)$$

Now, using (2.25), we obtain

$$\left| \frac{d}{dt} K_n(\lambda_x, \lambda_y) \right| \leq C|x| \left(n^{7/8} + |x-y| n^{-1} \right). \quad (2.71)$$

Finally, observing that

$$\frac{\partial}{\partial x} \mathcal{K}_n(x, y) + \frac{\partial}{\partial y} \mathcal{K}_n(x, y) = -(xn)^{-1} e^{-i(n-1)(x-y)/2n} \frac{d}{dt} K_n(\lambda_x, \lambda_y)|_{t=0},$$

$$\mathcal{K}_n(x, y) - \mathcal{K}_n(0, y-x) = e^{-i(n-1)(x-y)/2n} \cdot \frac{1}{n} \left(K_n(\lambda_x, \lambda_y)|_{t=0} - K_n(\lambda_x, \lambda_y)|_{t=1} \right),$$

and using (2.71), we conclude that the Lemma is proved. \blacksquare

Proof of Lemma 2.9. First show that for any $|x| \leq nd_0/2$ we have the bound

$$\int_{-1}^1 \frac{\mathcal{K}_n(x, x) \mathcal{K}_n(x+t, x+t) - |\mathcal{K}_n(x, x+t)|^2}{t^2} dt \leq C \quad (2.72)$$

Denote

$$\begin{aligned} \Omega_0 &= [-\pi + \lambda_0, \pi + \lambda_0], \quad \Omega_0^+ = \Omega_0 / \Omega_0^-, \\ \Omega_0^- &= \left\{ \lambda \in \Omega_0 : \left| \sin \frac{\lambda - \lambda_0}{2} \right| \leq \sin \frac{1}{2n} \right\} = [\lambda_0 - 1/n, \lambda_0 + 1/n], \end{aligned} \quad (2.73)$$

and consider the quantity

$$W = \left\langle \prod_{j=2}^n \left| 1 - \frac{\sin^2 1/2n}{\sin^2(\lambda_j - \lambda_0)/2} \right| \right\rangle, \quad (2.74)$$

where the symbol $\langle \dots \rangle$ denotes the average with respect to $p_n(\lambda_0, \lambda_2, \dots, \lambda_n)$. We will estimate W from above. To do this we use the relation

$$1 - \frac{\sin^2 \frac{1}{2n}}{\sin^2 \frac{\mu - \lambda}{2}} = \frac{(e^{i(\lambda+1/n)} - e^{i\mu})(e^{i(\lambda-1/n)} - e^{i\mu})}{(e^{i\lambda} - e^{i\mu})^2},$$

(1.2) and Schwarz inequality. We get that W^2 is not bigger than the product of two integrals I_+ and I_- , where

$$I_{\pm} = Z_n^{-1} \int_{\Omega_0^{n-1}} e^{-nV(\lambda_0)} \prod_{2 \leq j < k \leq n} |e^{i\lambda_j} - e^{i\lambda_k}|^2 \exp \left\{ -n \sum_{j=2}^n V(\lambda_j) \right\} \prod_{j=2}^n |e^{i(\lambda_0 \pm 1/n)} - e^{i\lambda_j}|^2 d\lambda_j.$$

Moreover, the expression $n(V(\lambda_0) - V(\lambda_0 \pm 1/n))$ is bounded in view of (1.17). Hence, from (1.15) we obtain

$$W \leq C \rho_n^{1/2} (\lambda_0 + 1/n) \rho_n^{1/2} (\lambda_0 - 1/n) \leq C. \quad (2.75)$$

On the other hand, W can be represent as follows

$$W = \left\langle \prod_{j=2}^n (\phi_1(\lambda_j) + \phi_2(\lambda_j)) \right\rangle, \quad (2.76)$$

where

$$\phi_1(\lambda) = \frac{\left(\sin^2 \frac{1}{2n} - \sin^2 \frac{\lambda - \lambda_0}{2} \right)^2}{\sin^2 \frac{1}{2n} \sin^2 \frac{\lambda - \lambda_0}{2}} \mathbf{1}_{\Omega_0^-}, \quad (2.77)$$

$$\phi_2(\lambda) = \left(1 - \frac{\sin^2 \frac{\lambda - \lambda_0}{2}}{\sin^2 \frac{1}{2n}} \right) \mathbf{1}_{\Omega_0^-} + \left(1 - \frac{\sin^2 \frac{1}{2n}}{\sin^2 \frac{\lambda - \lambda_0}{2}} \right) \mathbf{1}_{\Omega_0^+}. \quad (2.78)$$

Since $0 \leq \phi_2(\lambda) \leq 1$ and $\phi_1(\lambda) \geq 0$, it follows from (2.76) that W can be estimated bellow as

$$W \geq (n-1) \int_{\Omega_0} \phi_1(\lambda) \left\langle \delta(\lambda_2 - \lambda) \exp \left\{ \sum_{j=3}^n \ln \phi_2(\lambda_j) \right\} \right\rangle d\lambda.$$

Note that $\langle \delta(\lambda_2 - \lambda) \rangle = p_2^{(n)}(\lambda_0, \lambda)$. Therefore the Jensen inequality implies

$$\begin{aligned} W &\geq (n-1) \int_{\Omega_0^-} \phi_1(\lambda) p_2^{(n)}(\lambda_0, \lambda) \exp \left\{ \left\langle \delta(\lambda_2 - \lambda) \sum_{j=3}^n \ln \phi_2(\lambda_j) \right\rangle \left[p_2^{(n)}(\lambda_0, \lambda) \right]^{-1} \right\} d\lambda = \\ &= (n-1) \int_{\Omega_0^-} \phi_1(\lambda) p_2^{(n)}(\lambda_0, \lambda) \exp \left\{ (n-2) \int_{\Omega_0} \ln \phi_2(\lambda') p_3^{(n)}(\lambda_0, \lambda, \lambda') d\lambda' \left[p_2^{(n)}(\lambda_0, \lambda) \right]^{-1} \right\} d\lambda, \end{aligned}$$

where $p_k^{(n)}$ is defined in (1.5). Using (1.14) for $l = 2, 3$, we have

$$\begin{aligned} p_3^{(n)}(\lambda_0, \lambda, \lambda') &= \frac{n}{n-2} \rho_n(\lambda') p_2^{(n)}(\lambda_0, \lambda) + \\ &+ \frac{2\Re(K_n(\lambda_0, \lambda) K_n(\lambda, \lambda') K_n(\lambda', \lambda_0)) - K_n(\lambda_0, \lambda_0) |K_n(\lambda, \lambda')|^2 - K_n(\lambda, \lambda) |K_n(\lambda_0, \lambda')|^2}{n(n-1)(n-2)}. \end{aligned} \quad (2.79)$$

By the Cauchy inequality

$$\begin{aligned} 2 |K_n(\lambda_0, \lambda) K_n(\lambda, \lambda') K_n(\lambda', \lambda_0)| &\leq 2 K_n^{1/2}(\lambda_0, \lambda_0) K_n^{1/2}(\lambda, \lambda) |K_n(\lambda, \lambda') K_n(\lambda', \lambda_0)| \leq \\ &\leq K_n(\lambda_0, \lambda_0) |K_n(\lambda, \lambda')|^2 + K_n(\lambda, \lambda) |K_n(\lambda_0, \lambda')|^2, \end{aligned}$$

we obtain that the second term in (2.79) is non-positive, hence

$$p_3^{(n)}(\lambda_0, \lambda, \lambda') \leq \frac{n}{n-2} \rho_n(\lambda') p_2^{(n)}(\lambda_0, \lambda).$$

Taking into account that $\ln \phi_2(\lambda') \leq 0$, finally we get

$$W \geq (n-1) \int_{\Omega_0^-} \phi_1(\lambda) p_2^{(n)}(\lambda_0, \lambda) d\lambda \cdot \exp \left\{ n \int_{\Omega_0} \rho_n(\lambda') \ln \phi_2(\lambda') d\lambda' \right\}. \quad (2.80)$$

Now we will show that the second multiplier in (2.80) is bounded from below.

$$\begin{aligned}
n \int_{\Omega_0} \rho_n(\lambda') \ln \phi_2(\lambda') d\lambda' &= \\
&\left(\int_{|s| \leq 1} + \int_{1 \leq |s| \leq nd_0/2} + \int_{nd_0/2 \leq |s| \leq n\pi} \right) \rho_n(\lambda_0 + s/n) \ln \phi_2(\lambda_0 + s/n) ds \geq \\
&\geq C \left(\int_{|s| \leq 1} \ln \left(1 - \frac{\sin^2 s/(2n)}{\sin^2 1/(2n)} \right) ds + \int_{1 \leq |s| \leq nd_0/2} \ln \left(1 - \frac{\sin^2 1/(2n)}{\sin^2 s/(2n)} \right) ds \right) + \\
&\quad + \ln \left(1 - \frac{\sin^2 1/(2n)}{\sin^2 d_0/4} \right) \int_{|s| \leq n\pi} \rho_n(\lambda_0 + s/n) ds \geq C(I_1 + I_2) + O(n^{-1}). \\
I_1 &= \int_0^1 \ln \left(\frac{\cos(s/n) - \cos(1/n)}{1 - \cos(1/n)} \right) ds = -n \int_0^{1/n} \frac{\sin t}{\sin(t + 1/n)} \frac{t - 1/n}{2 \sin \frac{t - 1/n}{2}} dt \geq -C \\
I_2 &= n \int_{1/n}^{d_0/2} \ln \left(\frac{\cos(1/n) - \cos t}{1 - \cos t} \right) dt = \\
&= (nd_0/2 - 1) \ln \left(1 - \frac{\sin^2 1/2n}{\sin^2 d_0/2} \right) - n(1 - \cos 1/n) \int_{1/n}^{d_0/2} \cot t/2 \frac{t - 1/n}{2 \sin \frac{t - 1/n}{2}} \frac{1}{\sin \frac{t + 1/n}{2}} dt \geq \\
&\geq -C - Cn^{-1} \int_{1/n}^{d_0/2} \frac{dt}{t(t + 1/n)} \geq -C.
\end{aligned}$$

Thus, from (2.75) and (2.80) we obtain

$$n \int_{\Omega_0^-} \phi_1(\lambda) p_2^{(n)}(\lambda_0, \lambda) d\lambda \geq -C \quad (2.81)$$

Now, using (1.14), (2.27), (2.15), (2.77), and the inequality $\frac{1}{t^2} \leq C \frac{\sin^2 1/2n}{\sin^2 t/2n}$, we obtain (2.72) for $x = 0$ from (2.81). Changing λ_0 by $\lambda_0 + x/n$, we will get (2.72) for any $|x| \leq nd_0/2$.

Now we are ready to prove (2.36). Denote $C_n = \sup \left| \frac{\partial}{\partial x} \mathcal{K}_n(x, y) \right|$. In view of (2.32)

$$\begin{aligned}
C_n &\leq \left| \left(v.p. \int_{|z-x| \leq 1} + \int_{|z-x| \geq 1} \right) \frac{\mathcal{K}_n(x, z) \mathcal{K}_n(z, y)}{z - x} dz \right| + o(1) \leq \\
&\leq |I_1(x, y)| + |I_2(x, y)| + o(1).
\end{aligned}$$

Using the Schwarz inequality and (2.28) with (2.29), we can estimate I_2 as follows

$$|I_2(x, y)| \leq \mathcal{K}_n^{1/2}(x, x) \mathcal{K}_n^{1/2}(y, y) \leq C.$$

To estimate I_1 denote

$$\begin{aligned}\hat{t}_n^* &= \sup \{t > 0 : |x - y| \leq t \Rightarrow \mathcal{K}_n(x, y) \geq \rho_n(\lambda_0)/2\}, \\ t_n^* &= \min \{\hat{t}_n^*, 1\}.\end{aligned}\tag{2.82}$$

We will prove that the sequence t_n^* is bounded from below by some nonzero constant. Represent I_1 in the form

$$\begin{aligned}I_1(x, y) &= v.p. \int_{|t| \leq t_n^*} \frac{\mathcal{K}_n(x, x+t) \mathcal{K}_n(x+t, y) - \mathcal{K}_n(x, x) \mathcal{K}_n(x, y)}{t} dt + \\ &\quad + \int_{t_n^* \leq |t| \leq 1} \frac{\mathcal{K}_n(x, x+t) \mathcal{K}_n(x+t, y)}{t} dt = I_1' + I_1''.\end{aligned}$$

Using (2.29) we have $|I_1''| \leq C |\ln t_n^*|$. On the other hand, from (1.11) and Cauchy inequality we obtain for any x, y, z

$$\begin{aligned}|\mathcal{K}_n(x, z) - \mathcal{K}_n(y, z)|^2 &\leq (\mathcal{K}_n(x, x) + \mathcal{K}_n(y, y) - 2\mathcal{K}_n(x, y)) \mathcal{K}_n(z, z) = \\ &= \left(\left(\mathcal{K}_n^{1/2}(x, x) - \mathcal{K}_n^{1/2}(y, y) \right)^2 + 2 \left(\mathcal{K}_n^{1/2}(x, x) \mathcal{K}_n^{1/2}(y, y) - \mathcal{K}_n(x, y) \right) \right) \mathcal{K}_n(z, z).\end{aligned}\tag{2.83}$$

From (2.35) we get that the first term of (2.83) is bounded by $C n^{-1/4} |x - y|^2$. The second term we rewrite as

$$\mathcal{K}_n^{1/2}(x, x) \mathcal{K}_n^{1/2}(y, y) - \mathcal{K}_n(x, y) = \frac{\mathcal{K}_n(x, x) \mathcal{K}_n(y, y) - \mathcal{K}_n^2(x, y)}{\mathcal{K}_n^{1/2}(x, x) \mathcal{K}_n^{1/2}(y, y) + \mathcal{K}_n(x, y)}.$$

Thus, for $|x - y| \leq t_n^*$, we get

$$|\mathcal{K}_n(x, z) - \mathcal{K}_n(y, z)|^2 \leq C \left(n^{-1/4} |x - y|^{3/2} + \mathcal{K}_n(x, x) \mathcal{K}_n(y, y) - |\mathcal{K}_n(x, y)|^2 \right).\tag{2.84}$$

Hence, using (2.84), (2.72) and the Schwarz inequality, we obtain

$$|I_1'| \leq C \int_{|t| \leq t_n^*} \frac{|\mathcal{K}_n(x, x+t) - \mathcal{K}_n(x, x)| + |\mathcal{K}_n(x+t, y) - \mathcal{K}_n(x, y)|}{|t|} dt \leq C (t_n^*)^{1/2}.$$

Finally, from the above estimates we have

$$C_n \leq C \left(|\ln t_n^*| + (t_n^*)^{1/2} \right).\tag{2.85}$$

Note that if the sequence t_n^* is not bounded from below, then we have

$$C \leq \rho_n(\lambda_0)/2 \leq |\mathcal{K}_n(x + t_n^*, x) - \mathcal{K}_n(x, x)| \leq C_n t_n^* \leq C t_n^* \ln t_n^* + C t_n^*,$$

and we get a contradiction. Thus $t_n^* \geq d^*$, for some n -independent $d^* > 0$. Therefore from (2.85) we obtain the first inequality from (2.36).

To prove the second inequality of (2.36) we observe that we have by (2.33)

$$\int_{|x| \leq \mathcal{L}} \left| \frac{\partial}{\partial x} \mathcal{K}_n(x, y) \right|^2 dx = \int_{|x| \leq \mathcal{L}} \left| \frac{\partial}{\partial y} \mathcal{K}_n(x, y) \right|^2 dx + o(1).$$

Then we rewrite the analog of (2.32) for $\frac{\partial}{\partial y} \mathcal{K}_n(x, y)$ as

$$\begin{aligned} \frac{\partial}{\partial y} \mathcal{K}_n(x, y) &= \left(v.p. \int_{|z-y| \leq d^*} + \int_{|z| \leq 2\mathcal{L}} \mathbf{1}_{|z-y| \geq d^*} \right) \frac{\mathcal{K}_n(x, z) \mathcal{K}_n(z, y)}{y - z} + O(\mathcal{L}^{-1}) = \\ &= I_1(x, y) + I_2(x, y) + O(\mathcal{L}^{-1}). \end{aligned}$$

It is easy to see that it is enough to estimate $I_{1,2}^2$. Since in I_1 the domain of integration is symmetric with respect to y we can write

$$\begin{aligned} I_1(x, y) &= \int_{|z-y| \leq d^*} \frac{(\mathcal{K}_n(x, z) - \mathcal{K}_n(x, y)) \mathcal{K}_n(z, y)}{y - z} dz + \\ &+ \int_{|z-y| \leq d^*} \frac{(\mathcal{K}_n(z, y) - \mathcal{K}_n(y, y)) \mathcal{K}_n(x, y)}{y - z} dz. \end{aligned}$$

Now, using the Schwarz inequality and (2.28), we obtain

$$\begin{aligned} |I_1^2(x, y)| &\leq 2d^* C \int_{|z-y| \leq d^*} \frac{|\mathcal{K}_n(x, z) - \mathcal{K}_n(x, y)|^2}{(z - y)^2} dz + \\ &+ 2d^* \mathcal{K}_n^2(x, y) \int_{|z-y| \leq d^*} \frac{|\mathcal{K}_n(z, y) - \mathcal{K}_n(y, y)|^2}{(z - y)^2} dz. \end{aligned}$$

Integrating the above inequality with respect to x and using (2.28) with (2.29), we get

$$\begin{aligned} \int |I_1^2(x, y)| dx &\leq C \int_{|z-y| \leq d^*} \frac{|\mathcal{K}_n(z, y) - \mathcal{K}_n(y, y)|^2}{(z - y)^2} dz + \\ &+ C \int_{|z-y| \leq d^*} \frac{\mathcal{K}_n(z, z) + \mathcal{K}_n(y, y) - 2\mathcal{K}_n(z, y)}{(z - y)^2} dz. \end{aligned}$$

Now using bounds (2.83) in the second integral and (2.84) in the first, in view of (2.72) we obtain the bound for I_1^2 . To estimate I_2 we write

$$\begin{aligned} \int |I_2^2(x, y)| dx &\leq \int_{|z|, |z'| \leq 2\mathcal{L}} \mathbf{1}_{|z-y| > d^*} \mathbf{1}_{|z'-y| > d^*} \left| \frac{\mathcal{K}_n(y, z) \mathcal{K}_n(z, z') \mathcal{K}_n(z', y)}{(z - y)(z' - y)} \right| dz dz' \\ &\leq C \int_{|z|, |z'| \leq 2\mathcal{L}} \mathbf{1}_{|z-y| > d^*} \mathbf{1}_{|z'-y| > d^*} \left(\left| \frac{\mathcal{K}_n(y, z)}{z - y} \right|^2 + \left| \frac{\mathcal{K}_n(y, z')}{z' - y} \right|^2 \right) dz dz' \leq C. \end{aligned}$$

Above bounds for I_1 and I_2 prove the second inequality of (2.36). Thus, Lemma 2.9 is proved. \blacksquare

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